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Fixed Point Theorems and Convergence Theorems for Non-self Mappings in Hilbert Spaces (ヒルベルト空間における非自己写像の不動点定理と収束定理)

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Abstract. In this article, we first prove fixed point theorems for nonlinear non-self mappings in a Hilbert space. Next, we deal with weak and strong convergence theorems for nonlinear mappings in a Hilbert space. Using these results, we obtain new and well-known fixed point and convergence theorems. For example, we generalize Hojo and Takahashi's mean strong convergence theorem [11] for generalized hybrid mappings.

1 Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H . Kocourek, Takahashi and Yao [19] introduced a broad class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings, nonspreading mappings [21] and hybrid mappings [30]. A mapping $T : C \rightarrow H$ is said to be *generalized hybrid* [19] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \quad (1.1)$$

for all $x, y \in C$, where \mathbb{R} is the set of real numbers. We call such T an (α, β) -*generalized hybrid* mapping. An (α, β) -generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, i.e., $\|Tx - Ty\| \leq \|x - Ty\|$ for all $x, y \in C$. It is nonspreading for $\alpha = 2$ and $\beta = 1$, i.e., $2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2$ for all $x, y \in C$. Furthermore, it is hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e., $3\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 + \|y - x\|^2$ for all $x, y \in C$. They proved fixed point theorems and nonlinear ergodic theorems of Baillon's type [3] for generalized hybrid mappings in a Hilbert space; see also Kohsaka and Takahashi [20] and Iemoto and Takahashi [15]. Putting $x = u$ with $u = Tu$ in (1.1), we have that for any $y \in C$,

$$\alpha\|u - Ty\|^2 + (1 - \alpha)\|u - Ty\|^2 \leq \beta\|u - y\|^2 + (1 - \beta)\|u - y\|^2$$

and hence $\|u - Ty\| \leq \|u - y\|$. This means that an (α, β) -generalized hybrid mapping with a fixed point is quasi-nonexpansive. Kocourek, Takahashi and Yao [19] also introduced a more broad class of nonlinear mappings which covers generalized hybrid mappings. A mapping

$S : C \rightarrow H$ is called *super hybrid* [19, 34] if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} & \alpha \|Sx - Sy\|^2 + (1 - \alpha + \gamma) \|x - Sy\|^2 \\ & \leq (\beta + (\beta - \alpha)\gamma) \|Sx - y\|^2 + (1 - \beta - (\beta - \alpha - 1)\gamma) \|x - y\|^2 \\ & \quad + (\alpha - \beta)\gamma \|x - Sx\|^2 + \gamma \|y - Sy\|^2 \end{aligned} \quad (1.2)$$

for all $x, y \in C$. We call such a mapping an (α, β, γ) -*super hybrid* mapping. An $(\alpha, \beta, 0)$ -super hybrid mapping is (α, β) -generalized hybrid. So, the class of super hybrid mappings contains generalized hybrid mappings. On the other hand, Hojo, Takahashi and Yao [12] defined the following class of nonlinear mappings which contains generalized hybrid mappings. A mapping $U : C \rightarrow H$ is called *extended hybrid* if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} & \alpha(1 + \gamma) \|Ux - Uy\|^2 + (1 - \alpha(1 + \gamma)) \|x - Uy\|^2 \\ & \leq (\beta + \alpha\gamma) \|Ux - y\|^2 + (1 - (\beta + \alpha\gamma)) \|x - y\|^2 \\ & \quad - (\alpha - \beta)\gamma \|x - Ux\|^2 - \gamma \|y - Uy\|^2 \end{aligned} \quad (1.3)$$

for all $x, y \in C$. We note that super hybrid mappings and extended hybrid mappings are not quasi-nonexpansive generally. We also know the following relation between generalized hybrid mappings and extended hybrid mappings

Theorem 1.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \neq -1$. Let T and U be mappings of C into H such that $U = \frac{1}{1+\gamma}T + \frac{\gamma}{1+\gamma}I$, where $Ix = x$ for all $x \in H$. Then, for $1 + \gamma > 0$, $T : C \rightarrow H$ is an (α, β) -generalized hybrid mapping if and only if $U : C \rightarrow H$ is an (α, β, γ) -extended hybrid mapping.*

In this article, motivated by these mappings and results, we first prove fixed point theorems for nonlinear non-self mappings in a Hilbert space. Next, we deal with weak and strong convergence theorems for nonlinear mappings in a Hilbert space. Using these results, we obtain new and well-known fixed point and convergence theorems. For example, we generalize Hojo and Takahashi's mean strong convergence theorem [11] for generalized hybrid mappings.

2 Preliminaries

Throughout this paper, we denote by \mathbb{N} the set of positive integers. Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively. From [29], we know the following basic equality: For any $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda) \|x - y\|^2. \quad (2.1)$$

Furthermore, we know that for any $x, y, u, v \in H$

$$2 \langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \quad (2.2)$$

Let C be a nonempty closed convex subset of H and let T be a mapping from C into itself. Then, we denote by $F(T)$ the set of fixed points of T . A mapping $T : C \rightarrow H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \rightarrow H$ with $F(T) \neq \emptyset$

is called *quasi-nonexpansive* if $\|x - Ty\| \leq \|x - y\|$ for all $x \in F(T)$ and $y \in C$. Let C be a nonempty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. The mapping P_C is called the *metric projection* of H onto C . It is known that P_C is nonexpansive and $\langle x - P_C x, P_C x - u \rangle \geq 0$ for all $x \in H$ and $u \in C$. Furthermore, we know that

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle \quad (2.3)$$

for all $x, y \in H$; see [29] for more details. For proving main results in this paper, we also need the following lemmas proved in [31] and [2].

Lemma 2.1 ([31]). *Let D be a nonempty closed convex subset of H . Let P be the metric projection from H onto D . Let $\{u_n\}$ be a sequence in H . If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for all $u \in D$ and $n \in \mathbb{N}$, then $\{P u_n\}$ converges strongly to some $u_0 \in D$.*

Lemma 2.2 ([2]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n \rightarrow \infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \gamma_n + \beta_n$$

for all $n = 1, 2, \dots$. Then $\lim_{n \rightarrow \infty} s_n = 0$.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞ . If μ is a Banach limit on l^∞ , then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n \rightarrow \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \rightarrow \infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \rightarrow a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [27] for the proof of existence of a Banach limit and its other elementary properties. Using Banach limits, Kocourek, Takahashi and Yao [19] proved the following fixed point theorem for generalized hybrid mappings in a Hilbert space.

Theorem 2.3 ([19]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \rightarrow C$ be a generalized hybrid mapping. Then T has a fixed point in C if and only if $\{T^n z\}$ is bounded for some $z \in C$.*

3 Fixed Point Theorem for Non-Self Mappings

In this section, we first prove a fixed point theorem for generalized hybrid non-self mappings in a Hilbert space. For proving it, we need the following lemmas.

Lemma 3.1. *Let H be a Hilbert space and let C be a nonempty subset of H . Let α and β be in \mathbb{R} . Then, a non-self mapping $T : C \rightarrow H$ is (α, β) -generalized hybrid if and only if it satisfies that*

$$\|Tx - Ty\|^2 \leq (\alpha - \beta)\|x - y\|^2 + 2(\alpha - 1)\langle x - Tx, y - Ty \rangle - (\alpha - \beta - 1)\|y - Tx\|^2$$

for all $x, y \in C$.

Using Lemma 3.1, we have the following result.

Lemma 3.2. *Let H be a Hilbert space and let C be a nonempty bounded subset of H . If a non-self mapping $T : C \rightarrow H$ is generalized hybrid, then TC is bounded.*

The following is a fixed point theorem for non-self generalized hybrid mappings in a Hilbert space.

Theorem 3.3 ([12]). *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let α and β be real numbers. Let T be an (α, β) -generalized hybrid mapping with $\alpha - \beta \geq 0$ of C into H . Suppose that there exists $m > 1$ such that for any $x \in C$, $Tx = x + t(y - x)$ for some $y \in C$ and t with $1 \leq t \leq m$. Then, T has a fixed point in C .*

Recently, Hojo, Suzuki and Takahashi [10] also proved a more general fixed point theorem for nonlinear non-self mappings in a Hilbert space.

Theorem 3.4 ([10]). *Let C be a nonempty, bounded, closed and convex subset of a Hilbert space H and let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Let $T : C \rightarrow H$ be an $(\alpha, \beta, \gamma, \delta)$ -normal generalized hybrid mapping, i.e., there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that*

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0$$

for all $x, y \in C$. Suppose that it satisfies the following condition (1) or (2):

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ and $\alpha + \beta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and $\alpha + \gamma \geq 0$.

Assume that there exists $m > 1$ such that for any $x \in C$, $Tx = x + t(y - x)$ for some $y \in C$ and t with $0 < t \leq m$. Then T has a fixed point in C . In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) and (2).

For proving this result, Hojo, Suzuki and Takahashi [10] used the following fixed point theorem obtained by Kawasaki and Takahashi [18].

Theorem 3.5 ([18]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into itself, i.e., there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that*

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &+ \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0 \end{aligned}$$

for all $x, y \in C$. Suppose that it satisfies the following condition (1) or (2):

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ and $\zeta + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ and $\varepsilon + \eta \geq 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) and (2).

Let us give an example of mappings which is related to the conditions in Theorem 3.4. In the case of $H = \mathbb{R}$, consider a mapping $T : [0, 1] \rightarrow \mathbb{R}$:

$$Tx = (1 + 2x) \cos x - 2x^2, \quad \forall x \in [0, 1].$$

Then, we have

$$Tx = (1 + 2x)(\cos x - x) + x, \quad \forall x \in [0, 1].$$

Take $m = 3$. For any $x \in [0, 1]$, take $t = 1 + 2x$ and $y = \cos x$. Then, we have that $Tx = t(y - x) + x$, $y = \cos x \in [0, 1]$ and $0 < t = 1 + 2x \leq 3$.

4 Weak convergence theorems

In this section, using the technique developed by Takahashi [26], we first prove a mean convergence theorem of Baillon's type [3] for super hybrid mappings in a Hilbert space. For proving it, we need the following lemma.

Lemma 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let T be a generalized hybrid mapping from C into itself. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. Define $S_n x = \frac{1}{n} \sum_{k=1}^n T^k x$. Then, $\lim_{n \rightarrow \infty} \|S_n x - TS_n x\| = 0$. In particular, if C is bounded, then*

$$\lim_{n \rightarrow \infty} \sup_{x \in C} \|S_n x - TS_n x\| = 0.$$

Using Lemma 4.1, we obtain the the following mean convergence theorem.

Theorem 4.2 ([12]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let α, β and γ be real numbers with $\gamma \geq 0$ and let $S : C \rightarrow C$ be an (α, β, γ) -super hybrid mapping with $F(S) \neq \emptyset$ and let P be the metric projection of H onto $F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=1}^n \left(\frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I \right)^k x$$

converges weakly to $z \in F(S)$, where $z = \lim_{n \rightarrow \infty} PT^n x$ and $T = \frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I$.

Next, we prove a weak convergence theorem of Mann's type [23] for nonlinear non-self mappings in a Hilbert space. For proving the result, we need the following two lemmas.

Lemma 4.3. *Let C be a nonempty, closed and convex subset of a Hilbert space H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into H with $F(T) \neq \emptyset$ which satisfies the condition (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and $\zeta + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ and $\varepsilon + \eta \geq 0$.

Then T is quasi-nonexpansive.

We remark that if $T : C \rightarrow H$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [16]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \rightarrow z$. Since C is weakly closed, we have $z \in C$. Furthermore, from $\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \rightarrow 0$, we have that z is a fixed point of T and so $F(T)$ is closed. Let us show that $F(T)$ is convex.

For $x, y \in F(T)$ and $\alpha \in [0, 1]$, put $z = \alpha x + (1 - \alpha)y$. Then we have from (2.1) that

$$\begin{aligned}
 \|z - Tz\|^2 &= \|\alpha x + (1 - \alpha)y - Tz\|^2 \\
 &= \alpha\|x - Tz\|^2 + (1 - \alpha)\|y - Tz\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\
 &\leq \alpha\|x - z\|^2 + (1 - \alpha)\|y - z\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\
 &= \alpha(1 - \alpha)^2\|x - y\|^2 + (1 - \alpha)\alpha^2\|x - y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\
 &= \alpha(1 - \alpha)(1 - \alpha + \alpha - 1)\|x - y\|^2 = 0
 \end{aligned}$$

and hence $Tz = z$. This implies that $F(T)$ is convex.

Lemma 4.4. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $T : C \rightarrow H$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping. Suppose that it satisfies the following condition (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \gamma + \varepsilon + \eta > 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \beta + \zeta + \eta > 0$.

If $x_n \rightarrow z$ and $x_n - Tx_n \rightarrow 0$, then $z \in F(T)$.

Using Lemmas 4.3, 4.4 and the technique developed by Ibaraki and Takahashi [13, 14], we can prove the following weak convergence theorem.

Theorem 4.5 ([10]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $T : C \rightarrow H$ be a widely more generalized hybrid mapping with $F(T) \neq \emptyset$ which satisfies the condition (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ and $\varepsilon + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and $\zeta + \eta \geq 0$.

Let P be the metric projection of H onto $F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = P_C(\alpha_n x_n + (1 - \alpha_n)Tx_n), \quad n \in \mathbb{N}.$$

Then $\{x_n\}$ converges weakly to $v \in F(T)$, where $v = \lim_{n \rightarrow \infty} Px_n$.

Using Theorem 4.5, we can show the following weak convergence theorem of Mann's type for generalized hybrid mappings in a Hilbert space.

Theorem 4.6 ([19]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $T : C \rightarrow C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges weakly to an element $v \in F(T)$.

Proof. Since $T : C \rightarrow C$ is a generalized hybrid mapping, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - Ty\|^2 + (1 - \beta)\|x - Ty\|^2$$

for all $x, y \in C$. We have that this mapping is an $(\alpha, 1 - \alpha, -\beta, -(1 - \beta), 0, 0, 0)$ -widely more generalized hybrid mapping which satisfies the condition (2) in Theorem 4.5. Therefore, we have the desired result from Theorem 4.5. \square

5 Strong Convergence Theorem

In this section, using an idea of mean convergence by Shimizu and Takahashi [24] and [25], we prove a strong convergence theorem of Halpern's type for super hybrid mappings in a Hilbert space.

Theorem 5.1 ([12]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let α, β and γ be real numbers with $\gamma \geq 0$. Let $S : C \rightarrow C$ be a (α, β, γ) -super hybrid mapping with $F(S) \neq \emptyset$ and let P be the metric projection of H onto $F(S)$. Suppose that $\{x_n\}$ is a sequence generated by $x_1 = x \in C, u \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n, \\ z_n = \frac{1}{n} \sum_{k=1}^n \left(\frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I \right)^k x_n \end{cases}$$

for all $n = 1, 2, \dots$, where $0 \leq \alpha_n \leq 1, \alpha_n \rightarrow 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to Pu .

Recently, Hojo, Suzuki and Takahashi [10] also proved the following strong convergence theorem for widely more generalized hybrid mappings in a Hilbert space.

Theorem 5.2 ([10]). *Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let T be a widely more generalized hybrid mapping of C into itself which satisfies the following condition (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0, \alpha + \gamma > 0, \varepsilon + \eta \geq 0$ and $\zeta + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0, \alpha + \beta > 0, \zeta + \eta \geq 0$ and $\varepsilon + \eta \geq 0$.

Let $u \in C$ and define sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n, \\ z_n = \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n = 1, 2, \dots$, where $0 \leq \alpha_n \leq 1, \alpha_n \rightarrow 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T) \neq \emptyset$, then $\{x_n\}$ and $\{z_n\}$ converge strongly to Pu , where P is the metric projection of H onto $F(T)$.

Using Theorem 5.2, we can show the following result obtained by Hojo and Takahashi [11].

Theorem 5.3 ([11]). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let T be a generalized hybrid mapping of C into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n, \\ z_n = \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n = 1, 2, \dots$, where $0 \leq \alpha_n \leq 1, \alpha_n \rightarrow 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T)$ is nonempty, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $Pu \in F(T)$, where P is the metric projection of H onto $F(T)$.

Proof. As in the proof of Theorem 4.6, a generalized hybrid mapping is a widely more generalized hybrid mapping. Therefore, we have the desired result from Theorem 5.2. \square

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